

UNIQUENESS OF GABOR SERIES

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ABSTRACT. We prove that any complete and minimal Gabor system of Gaussians is a Markushevich basis in $L^2(\mathbb{R})$.

1. INTRODUCTION

Let $\Lambda \subset \mathbb{R}^2$ be a sequence of distinct points. With each such sequence we associate Gabor system

$$(1.1) \quad \mathcal{G}_\Lambda := \{e^{2\pi i y t} e^{-\pi(t-x)^2}\}_{(x,y) \in \Lambda}.$$

Function $e^{2\pi i y t} e^{-\pi(t-x)^2}$ can be viewed as the time–frequency shift of the Gaussian $e^{-\pi t^2}$ in the phase space. It is well known that system \mathcal{G}_Λ cannot be a Riesz basis in $L^2(\mathbb{R})$ (see e.g [9]). On the other hand, there exist a lot of *complete and minimal* systems \mathcal{G}_Λ . A canonical example is the lattice without one point, $\Lambda := \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$. However, the generating sets Λ can be very far from any lattice. For example, in [1] it was shown that there exists $\Lambda \subset \mathbb{R} \times \{0\} \cup \{0\} \times \mathbb{R}$ such that \mathcal{G}_Λ is complete and minimal in $L^2(\mathbb{R})$.

If \mathcal{G}_Λ is complete and minimal, then there exists the unique biorthogonal system $\{g_{(x,y)}\}_{(x,y) \in \Lambda}$. So, for any $f \in L^2(\mathbb{R})$ we may write the formal Fourier series with respect to the system \mathcal{G}_Λ

$$(1.2) \quad f \sim \sum_{(x,y) \in \Lambda} (f, g_{(x,y)})_{L^2(\mathbb{R})} e^{2\pi i y t} e^{-\pi(t-x)^2}.$$

If $\Lambda = \mathbb{Z} \times \mathbb{Z} \setminus \{(0,0)\}$, then it is known that there exists a linear summation method for the series (1.2) (e.g. one can use methods from [8]). In [8] this was proved for certain sequences similar to lattices. The main point of the present note is to show that *any* series (1.2) defines an element f uniquely.

Theorem 1.1. *Let \mathcal{G}_Λ be a complete and minimal system in $L^2(\mathbb{R})$. Then the biorthogonal system $\{g_{(x,y)}\}_{(x,y) \in \Lambda}$ is complete. So, any function $f \in L^2(\mathbb{R})$ is uniquely determined by the coefficients $(f, g_{(x,y)})$.*

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This property is by no means automatic for an arbitrary system of vectors. Indeed, if $\{e_n\}_{n=1}^\infty$ is an orthonormal basis in a separable Hilbert space, then $\{e_1 + e_n\}_{n=2}^\infty$ is a complete and minimal system but its biorthogonal $\{e_n\}_{n=2}^\infty$ is not complete. A complete and minimal system in a Hilbert space with complete biorthogonal system is called *Markushevich basis*.

Theorem 1.1 is analogous to Young's theorem [11] for systems of complex exponentials $\{e^{i\lambda_n t}\}$ in L^2 of an interval. However, the structure of complete and minimal systems for Gabor systems is more puzzling than for the systems of exponentials on an interval. For example, if Λ generates a complete and minimal system of exponentials in $L^2(-\pi, \pi)$, then the upper density of Λ ($= \lim_{r \rightarrow \infty} \#(\Lambda \cap \{|\lambda| < r\})(2r)^{-1}$) is equal to 1; see Theorem 1 in Lecture 17 of [7]. On the other hand, if \mathcal{G}_Λ is a complete and minimal Gabor system, then the upper density of Λ ($= \lim_{r \rightarrow \infty} \#(\Lambda \cap \{x^2 + y^2 \leq r^2\})(\pi r^2)^{-1}$) can vary from $2/\pi$ to 1; see Theorem 1 in [1]. If, in addition, Λ is a regular distributed set, then the upper density have to be from $2/\pi$ to 1; see Theorem 2 in [1].

Note that for some systems of special functions (associated to some canonical system of differential equations) in L^2 of an interval completeness of biorthogonal system may fail (even with infinite defect); see [2, Proposition 3.4].

In the next section we transfer our problem to Fock space of entire functions. The last section is devoted to the proof of our result.

Notations. Throughout this paper the notation $U(x) \lesssim V(x)$ means that there is a constant C such that $U(x) \leq CV(x)$ holds for all x in the set in question, $U, V \geq 0$. We write $U(x) \asymp V(x)$ if both $U(x) \lesssim V(x)$ and $V(x) \lesssim U(x)$.

2. REDUCTION TO A FOCK SPACE PROBLEM

Let

$$\mathcal{F} := \{F \text{ is entire and } \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) < \infty\};$$

here dm denotes the planar Lebesgue measure. It is well known that the following Bargmann transform

$$\begin{aligned} \mathcal{B}f(z) &:= 2^{1/4} e^{-i\pi xy} e^{\frac{\pi}{2}|z|^2} \int_{\mathbb{R}} f(t) e^{2\pi i y t} e^{-\pi(t-x)^2} dt \\ &= 2^{1/4} \int_{\mathbb{R}} f(t) e^{-\pi t^2} e^{2\pi t z} e^{-\frac{\pi}{2} z^2} dt, \quad z = x + iy, \end{aligned}$$

is a unitary map between $L^2(\mathbb{R})$ and the Fock space \mathcal{F} ; see [5, 6] for the details.

Moreover, the time–frequency shift of the Gaussian is mapped to the normalized reproducing kernel of \mathcal{F}

$$(2.1) \quad 2^{1/4} \mathcal{B}(e^{2\pi i u t} e^{-\pi(t-v)^2})(z) = e^{-\pi|w|^2/2} e^{\pi \bar{w} z} = \frac{k_w(z)}{\|k_w\|_{\mathcal{F}}}, \quad w = u - iv, \quad k_w(z) := e^{\pi \bar{w} z}.$$

The existence of such transformation allows us to apply methods from the theory of entire functions. For that reason the results about time–frequency shifts of the Gaussians are stronger than for the time–frequency shifts of other elements of $L^2(\mathbb{R})$.

Lemma 2.1. *The system \mathcal{G}_{Λ} is complete and minimal in $L^2(\mathbb{R})$ if and only if the system of reproducing kernels $\left\{ \frac{k_{\lambda}(z)}{\|k_{\lambda}\|} \right\}_{\lambda \in \Lambda}$ is complete and minimal in \mathcal{F} .*

Proof. The system \mathcal{G}_{Λ} is complete and minimal if and only if the system $\mathcal{G}_{\bar{\Lambda}}$ is complete and minimal. Now Lemma 2.1 immediately follows from the unitarity of Bargmann transform. \square

In many spaces of entire functions the system biorthogonal to the system of reproducing kernels can be described via the generating function; see e.g. Theorem 4 in Lecture 18 of [7] (this idea goes back to Paley and Wiener).

Lemma 2.2. *The system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is complete and minimal in \mathcal{F} if and only if there exists an entire function F such that F has simple zeros exactly at Λ , $\frac{F(z)}{z-\lambda}$ belongs to \mathcal{F} for some (any) $\lambda \in \Lambda$ and there is no non-trivial entire function T such that $FT \in \mathcal{F}$.*

Proof. Necessity. The system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ has a biorthogonal system which we will call $\{F_{\lambda}\}_{\lambda \in \Lambda}$. We know that $F_{\lambda_1}(z) \frac{z-\lambda_1}{z-\lambda_2} \in \mathcal{F}$ for any $\lambda_1, \lambda_2 \in \Lambda$. This function vanishes at the points $\lambda \in \Lambda \setminus \{\lambda_2\}$ and so it equals F_{λ_2} up to a multiplicative constant. Hence, the function $c_{\lambda} F_{\lambda}(z)(z-\lambda)$ does not depend on λ for suitable coefficients c_{λ} . Denote it by F . It is easy to see that F satisfies the required properties.

Sufficiency. Assume that such F exists. From the inclusion $\frac{F(z)}{z-\lambda_0} \in \mathcal{F}$ we conclude that the system $\{k_{\lambda}\}_{\lambda \in \Lambda \setminus \{\lambda_0\}}$ is not complete. On the other hand, if the whole system $\{k_{\lambda}\}_{\lambda \in \Lambda}$ is not complete, then there exists T such that $FT \in \mathcal{F}$. \square

The function F from Lemma 2.2 is called *a generating function* of Λ . So, the following theorem is the reformulation of Theorem 1.1 in terms of the Fock space.

Theorem 2.3. *If $\{k_{\lambda}\}$ is a complete and minimal system of reproducing kernels in \mathcal{F} and F is the generating function of this system, then the system $\left\{ \frac{F(z)}{z-\lambda} \right\}_{\lambda \in \Lambda}$ is also complete.*

In the last section we will prove this theorem.

3. COMPLETENESS OF BIORTHOGONAL SYSTEM

3.1. Preliminary steps. Let σ be the Weierstrass σ -function associated to the lattice $\mathcal{Z} = \{z : z = m + in, m, n \in \mathbb{Z}\}$,

$$\sigma(z) = z \prod_{\lambda \in \mathcal{Z} \setminus \{0\}} \left(1 - \frac{z}{\lambda}\right) e^{\frac{z}{\lambda} + \frac{z^2}{\lambda^2}}.$$

It is well known that $|\sigma(z)| \asymp \text{dist}(z, \mathcal{Z})e^{\pi|z|^2/2}$; see e.g. [10, p. 108]. From this estimate it is easy to see that system $\left\{\frac{k_w}{\|k_w\|}\right\}_{w \in \mathcal{Z} \setminus \{0\}}$ is a complete and minimal system and $\sigma_0(z) := \frac{\sigma(z)}{z}$ is its generating function. The system $\left\{\frac{\|k_w\|}{\sigma'_0(w)} \cdot \frac{\sigma_0(z)}{z-w}\right\}$ is the biorthogonal system. With any function $S \in \mathcal{F}$ we can associate its formal Fourier series *with respect to the system* $\left\{\frac{k_w}{\|k_w\|}\right\}_{w \in \mathcal{Z} \setminus \{0\}}$

$$S \sim \sum_{w \in \mathcal{Z} \setminus \{0\}} b_w \frac{k_w}{\|k_w\|}, \quad b_w := \left(S(z), \frac{\|k_w\|}{\sigma'_0(w)} \cdot \frac{\sigma_0(z)}{z-w}\right)_{\mathcal{F}}.$$

This series is more regular than an arbitrary Fourier series (1.2). For example this series admits a linear summation method. In particular, we know that the sequence $\{b_w\}$ is non-trivial. We need the following straightforward estimate of coefficients

$$\begin{aligned} |b_w|^2 &\leq \|S\|^2 \cdot \left\|\frac{\|k_w\|}{\sigma'_0(w)} \cdot \frac{\sigma_0(z)}{z-w}\right\|^2 \lesssim \|S\|^2 \cdot \left\|\frac{w\sigma_0(z)}{z-w}\right\|^2 \\ (3.1) \quad &\lesssim \|S\|^2 \cdot \left[\int_{|z| < 2|w|} |\sigma_0^2(z)| e^{-\pi|z|^2} dm(z) + 1\right] \lesssim \|S\|^2 \cdot \log(1 + |w|). \end{aligned}$$

Lemma 3.1. *If F is the generating function of a complete and minimal system of reproducing kernels $\{k_\lambda\}_{\lambda \in \Lambda}$ in \mathcal{F} and $\Lambda \cap \mathcal{Z} = \emptyset$, then for any triple $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ we have*

$$(3.2) \quad \left(\frac{F(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}, S\right)_{\mathcal{F}} = \sum_{w \in \mathcal{Z} \setminus \{0\}} \frac{F(w)b_w}{(w-\lambda_1)(w-\lambda_2)(w-\lambda_3)\|k_w\|}$$

for any $S \in \mathcal{F}$.

Proof. It is well known that for any function $H \in \mathcal{F}$ we have $\sum_{w \in \mathcal{Z} \setminus \{0\}} \frac{|H(w)|^2}{\|k_w\|^2} < \infty$ (see e.g. [4]). So, $\left\{\frac{F(w)}{(w-\lambda_1)\|k_w\|}\right\} \in \ell^2$. From (3.1) we conclude that $\left\{\frac{b_w}{(w-\lambda_2)(w-\lambda_3)}\right\} \in \ell^2$. Hence, the series on the right hand side of (3.2) converges and defines a bounded linear functional on \mathcal{F} . On the other hand, the left hand side and the right hand side of (3.2) coincides if S is a finite linear combination of $\{k_w\}_{w \in \mathcal{Z} \setminus \{0\}}$. \square

3.2. Proof of Theorem 1.1. Assume the contrary. Then there exists a function $S \in \mathcal{F}$ such that $S \perp \frac{F(z)}{z-\lambda}$ for any $\lambda \in \Lambda$. Without loss of generality we can assume that $\Lambda \cap \mathcal{Z} = \emptyset$. From the identity

$$\frac{1}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)} = \sum_{k=1}^3 \frac{c_k}{z-\lambda_k}$$

we get that for any triple $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$

$$\left(\frac{F(z)}{(z-\lambda_1)(z-\lambda_2)(z-\lambda_3)}, S \right) = 0.$$

Fix two arbitrary points $\lambda_1, \lambda_2 \in \Lambda$. Put

$$L(z) = \sum_{w \in \mathcal{Z} \setminus \{0\}} \frac{F(w)\overline{b_w}}{(z-w)(w-\lambda_1)(w-\lambda_2)\|k_w\|}.$$

Using Lemma 3.1 we get that meromorphic function L vanishes at $\Lambda \setminus \{\lambda_1, \lambda_2\}$. Hence,

$$(3.3) \quad \sum_{w \in \mathcal{Z} \setminus \{0\}} \frac{F(w)\overline{b_w}}{(z-w)(w-\lambda_1)(w-\lambda_2)\|k_w\|} = \frac{F(z)T(z)}{(z-\lambda_1)(z-\lambda_2)\sigma_0(z)},$$

where T is some non-zero entire function. Comparing the residues of both sides of (3.3) we get $T(w) = \overline{b_w} \frac{\sigma'_0(w)}{\|k_w\|}$, $w \in \mathcal{Z} \setminus \{0\}$. Assume that T has at least two zeros t_1, t_2 , then

$$(3.4) \quad F(z) \frac{T(z)}{(z-t_1)(z-t_2)} = \sum_{w \in \mathcal{Z} \setminus \{0\}} \frac{|w|^{1/2}\sigma_0(z)}{z-w} \cdot \frac{F(w)\overline{b_w}}{(w-t_1)(w-t_2)|w|^{1/2}\|k_w\|}.$$

From the inclusion $\left\{ \frac{F(w)}{(w-\lambda_1)\|k_w\|} \right\} \in \ell^2$ and estimates $|b_w|^2 \lesssim \log(1+|w|)$, $\left\| \frac{|w|^{1/2}\sigma_0(z)}{z-w} \right\| \lesssim 1$ we get that the right hand side of (3.4) belongs to \mathcal{F} . This contradicts the completeness of sequence $\{k_\lambda\}_{\lambda \in \Lambda}$.

Hence T has at most one zero. So, $T(z) = e^{P(z)}(a_1z - a_0)$, where P is a polynomial of degree at most 2. This contradicts the estimate $|T(w)| = |b_w \frac{\sigma'_0(w)}{\|k_w\|}| \lesssim \frac{\log^{1/2}(1+|w|)}{|w|}$, $w \in \mathcal{Z} \setminus \{0\}$. \square

3.3. Concluding remarks. 1. The author wonders if the following statement (stronger than Theorem 1.1) is true:

Question 1. *Any complete and minimal Gabor system is a strong Markushevich basis. Which means that any vector $f \in L^2(\mathbb{R})$ belongs to the closed linear span of members of its Fourier series (1.2) (see [3] and references therein).*

For systems of complex exponentials $\{e^{i\lambda_n t}\}$ in L^2 of an interval this is not true; see [3, Theorem 2].

2. Using our methods one can prove the completeness of the system $\{\frac{F(z)}{z-\lambda}\}_{\lambda \in \Lambda}$ under weaker assumptions than in Theorem 2.3 (e.g if $F \in \mathcal{F}$ and $z^n F \notin \mathcal{F}$, $n \in \mathbb{N}$). Nevertheless we prefer to formulate the result as it is to avoid inessential technicalities.

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